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| Benha University<br>Faculty of Engineering- Shoubra<br>Eng. Mathematics & Physics Department<br>Preparatory Year | Final Term Exam<br>Date: January 9 , 2018<br>Course: Mathematics 1 – A<br>Duration: **3** hours |

The Exam consists of one page   Answer **All** Questions   No. of questions: 5   Total Mark: 100

**Question 1**

(a)Complete the following : 5

(i)The eigenvectors of a symmetric matrix of real numbers are ……………………….

(ii)A square matrix A is called singular if … and it is called symmetric if …………

(iii)If $\lambda_1, \lambda_2, \dots, \lambda_n$ are all eigenvalues of a matrix A, then trace A is…………………

(iv)A linear system AX = B is consistent if ………………………………………..

(v)A linear system AX = 0 has infinite number of solutions if |A| is…………………

(b)Find the eigenvalues and the eigenvectors of the matrix : $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ 10

Also, Write the Hamilton equation, the diagonal form and find $f(A) = 2^A$.

**Question 2**

(a) If $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$ 10

Find, if possible A + B , A + C, A.B , A.C , C.A , |A| , |A`.A|

(b)Write the matrix of the L.T : $AX = \begin{bmatrix} 2x + y \\ x - 3y \end{bmatrix}$ and find its Kernel. 5

(c)Solve the L.S : (i) $3x - y + z = 3,\quad 4x + 2z = 5,\quad x + y + z = 2$ 4

(ii) $x - 2y + 2z = 2,\quad 2x - z = -3,\quad 3x - y + z = 1$ 4

**Question 3**

(a)Find $S_n$ *and* $S_8$ from the series : $\sum_{r=1}^{n} (2r + 1)(r + 2)$ 5

(b)By Induction, prove that : $1.3 + 2.4 + 3.5 + \cdots + n(n + 2) = \frac{1}{6} n(n + 1)(2n + 7)$ 5

(c)If $z_1 = 3 - 2i,\quad z_2 = -1 - i$. Find $z_1 + z_2,\ z_1 - z_2,\ z_1.z_2,\ \frac{z_1}{z_2},\ (z_2)^6$. 5

**Question 4**

(a) If $\cos(xt) + e^{\tan 5t} = 3x^2$ , $\ln(y^2\, t) + \sec t^3 = y^{\cot y}$. Find $y\grave{} = \frac{dy}{dx}$. 6

(b)Find the equations of tangent and normal lines to the curve : $y = \ln x^{\cos x}$ at $x = \frac{\pi}{2}$ 6

(c)Find $y^{(n)}$ where : (i) $y = x.(\cos 3x)^2 . \sin 2x$   (ii) $y = \ln \sqrt[7]{\frac{15+2x-x^2}{16-x^2}}$ 8

**Question 5**

(a)Find the limits :

(i) $\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$   (ii) $\lim_{x \to 0} \left(\frac{1}{\ln(x + 1)} - \frac{1}{e^x - 1}\right)$   (iii) $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$ 9

(b)Write the Maclaurin's series of the function : $f(x) = x^3 \ln(x + 1)$. 6

(c)Determine the value of c that satisfies the mean value theorem for the function : 6

$f(x) = x^3 + 9x^2 + 13x$ in the interval [–7 , – 1].

(d)Find the integrals : (i) $\int \frac{1}{x} (2 + \frac{3}{x})^{-1} dx$   (ii) $\int \frac{\sec^2 3x}{\sqrt{1+\tan 3x}} dx$ 6

*Good Luck*   *Dr. Kalled Elnagar*   *Dr. Ibrahim Sakr*   *Dr. Mohamed Eid*

# *Model Answer*

**Answer of Question 1**

(a)(i)The eigenvectors of a symmetric matrix of real numbers are orthogonal.

(ii)A square matrix A is called singular if |A| = 0 and it is called symmetric if A\` = A.

(iii)If $\lambda_1, \lambda_2, \dots, \lambda_n$ are all eigenvalues of a matrix A, then trace A is $\lambda_1 + \lambda_2 + \cdots + \lambda_n$

(iv)A linear system AX = B is consistent if it has a solution.

(v)A linear system AX = 0 has infinite number of solutions if |A| is zero.

-------------------------------------------------------------------------------------------------*5 Marks*

(b) $\begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) = \lambda^2 - 5\lambda + 6 = 0$. Then $\lambda_1 = 2,\quad \lambda_2 = 3$

From the equation, $\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

For : $\lambda_1 = 2$ , $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then 0x + y = 0, 0x + y = 0

Then y = 0, x = any number except 0

Put x = 1, then the corresponding eigenvector is : $X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

For : $\lambda_2 = 3$ , $\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

Then –x + y = 0, 0x + 0y = 0

Then x = y = any number except 0

Put y = 1, we get x = 1 and the corresponding eigenvector is: $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

The Hamilton's equation : $A^2 - 5A + 6I = 0$.

The diagonal form : $T^{-1}A\,T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $T^{-1}2^A\,T = \begin{bmatrix} 2^2 & 0 \\ 0 & 2^3 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}$

Then $f(A) = 2^A = T\begin{bmatrix} 4 & 0 \\ 0 & 8 \end{bmatrix}T^{-1} = \begin{bmatrix} 4 & 4 \\ 0 & 8 \end{bmatrix}$

-----------------------------------------------------------------------------------------------*10 Marks*

**Answer of Question 2**

(a) $A + C$, A.B, A.C, |A| are not exist.

$A + B = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 3 \end{bmatrix}$, $C.A = \begin{bmatrix} 7 & -5 & 6 \\ 4 & -5 & 7 \end{bmatrix}$, $A\grave{}.A = \begin{bmatrix} 5 & -4 & 5 \\ -4 & 5 & -7 \\ 5 & -7 & 10 \end{bmatrix}$, $|A\grave{}.A| = 0$

-----------------------------------------------------------------------------------------------*10 Marks*

(b)The matrix of transformation is : $A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

From the equations : $2x + y = 0$, $x - 3y = 0$, we get x = y = 0.

Then $\text{Ker } A = \left\{X \in R^2 : X = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$

-------------------------------------------------------------------------------------------------*5 Marks* (c)(i) $G = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 3 & -1 & 1 & 3 \\ 4 & 0 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & -4 & -2 & -3 \\ 0 & -4 & -2 & -3 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 2 \\ 0 & -4 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Rank A = 2 = Rank G. Infinite number of solutions.

Putting z = 0, we get $y = \frac{3}{4}$ and $x = \frac{5}{4}$

(ii)By Calculator, the solution is : (0, 2, 3).

-----------------------------------------------------------------------------------------*8 Marks*

**<u>Answer of Question 3</u>**

(a)Since $u_r = (2r + 1)(r + 2) = 2r^2 + 5r + 2$

Then $S_n = \frac{2}{6} n(n + 1)(2n + 1) + \frac{5}{2} n(n + 1) + 2n$

Then $S_8 = \frac{1}{3}(8)(9)(17) + (20)(9) + 16 = 604$

-----------------------------------------------------------------------------------------*5 Marks*

(b)(1)At n = 1, the left hand side (L.H.S) of this relation is 3 and the right hand side (R.H.S) of this relation is 3. Then this relation is true at n = 1.

(2)Assume that this relation is true at n = k.

This means that $1.3 + 2.4 + 3.5 + \cdots + k(k + 2) = \frac{1}{6} k(k + 1)(2k + 7)$

(3)We shall prove that this relation is true at n = k + 1.

From step 2, add (k + 1)(k + 3) to both sides, we get

$$1.3 + 2.4 + 3.5 + \cdots + k(k + 2) + (k + 1)(k + 3) = \frac{1}{6} k(k + 1)(2k + 7) + (k + 1)(k + 3)$$

$$= \frac{1}{6}(k + 1)[k(2k + 7) + 6k + 18]$$

$$= \frac{1}{6}(k + 1)(k + 2)(2k + 9)$$

Then this relation is true for all $n \geq 1$.

-----------------------------------------------------------------------------------------*5 Marks*

(c) $z_1 + z_2 = 2 - 3i, \quad z_1 - z_2 = 4 - i, \quad z_1 . z_2 = -5 - i$

$$\frac{z_1}{z_2} = \frac{3 - 2i}{-1 - i} . \frac{-1 + i}{-1 + i} = \frac{-1 + 5i}{2} = -\frac{1}{2} + \frac{5}{2} i$$

Since $z_2 = -1 - i = \sqrt{2}\, e^{\frac{i5\pi}{4}}$. Then $(z_2)^6 = 2^3\, e^{i\frac{30\pi}{4}} = 8\, e^{i\frac{3\pi}{2}} = -8i$

-----------------------------------------------------------------------------------------*5 Marks*

*Dr. Mohamed Eid*

**Answer of Question 4**

a) $\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}$, where $-\sin(x\,t)\,[x\frac{dt}{dx}+t]+5\sec^2(5t)\,e^{\tan(5t)}\frac{dt}{dx}=6x$

$$\Rightarrow \frac{dt}{dx}=\frac{6x+t\sin(x\,t)}{5\sec^2(5t)\,e^{\tan 5t}-x\sin(x\,t)}$$

And $\dfrac{y^2+2y\,t\frac{dy}{dt}}{y^2\,t}+3t^2\sec(t^3)\tan(t^3)=y^{\cot(y)}[-\csc^2(y)\,\ell n\,y+\frac{1}{y}\cot y]\frac{dy}{dt}$

$$\Rightarrow y^2+2y\,t\frac{dy}{dt}+3t^3y^2\sec(t^3)\tan(t^3)=y^2\,t\,y^{\cot(y)}[-\csc^2(y)\,\ell n\,y+\frac{1}{y}\cot y]\frac{dy}{dt}$$

$$\Rightarrow y^2+3t^3y^2\sec(t^3)\tan(t^3)=(y^2\,t\,y^{\cot(y)}[-\csc^2(y)\,\ell n\,y+\frac{1}{y}\cot y]-2\,y\,t)\frac{dy}{dt}$$

$$\Rightarrow \frac{dy}{dt}=\frac{y^2+3t^3y^2\sec(t^3)\tan(t^3)}{(y^2\,t\,y^{\cot(y)}[-\csc^2(y)\,\ell n\,y+\frac{1}{y}\cot y]-2\,y\,t)}$$

b) The slope of the tangent to the curve $y=\ell n\,x^{\cos x}$ at $x=\frac{\pi}{2}$ is expressed by $y`(\frac{\pi}{2})$, where $y` = \cos x\,(\frac{1}{x}) - \sin x\ \ell n\,x$, therefore $y`(\frac{\pi}{2}) = -\ \ell n\,(\frac{\pi}{2})$ and hence the slope of the normal $=\dfrac{1}{\ell n(\frac{\pi}{2})}$.

At $x=\frac{\pi}{2}$, $y=\ell n\,(\frac{\pi}{2})^0=0$, hence the point of contact is $(\frac{\pi}{2},0)$.

Therefore the equation of tangent is $\dfrac{y-0}{x-\frac{\pi}{2}}=-\ell n(\frac{\pi}{2})$ and the equation of normal is

$$\frac{y-0}{x-\frac{\pi}{2}}=\frac{1}{\ell n(\frac{\pi}{2})}.$$

c-i) $y = x\cos^2(3x)\ \sin(2x) = \frac{x}{2}(1+\cos 6x)\sin 2x = x[\frac{1}{2}\sin 2x+\frac{1}{4}(-\sin 4x+\sin 8x)]$

Therefore $y^{(n)}=\frac{1}{2}[2^n\sin(2x+\frac{n\pi}{2})+\frac{1}{2}(-4^n\sin(4x+\frac{n\pi}{2})+8^n\sin(8x+\frac{n\pi}{2}))]x\ +$

$$\frac{n}{2}[2^{n-1}\sin(2x+\frac{(n-1)\pi}{2})+\frac{1}{2}(-4^{n-1}\sin(4x+\frac{(n-1)\pi}{2})+8^{n-1}\sin(8x+\frac{(n-1)\pi}{2}))]$$

c-ii) $g(x)=\ell n[\sqrt[7]{\frac{15+2x-x^2}{16-x^2}}]=\frac{1}{7}[\ell n(5-x)(x+3)-\ell n(4-x)(4+x)]=$

$\frac{1}{7}[\ell n(5-x)+\ell n(x+3)-\ell n(4-x)-\ell n(4+x)]$, therefore

$$y^{(n)}(x) = \frac{1}{7}\left[\frac{(-1)^{n-1}(n-1)!(-1)^n}{(5-x)^n}+\frac{(-1)^{n-1}(n-1)!}{(x+3)^n}-\frac{(n-1)!(-1)^n}{(4-x)^n}-\frac{(-1)^{n-1}(n-1)!}{(4+x)^n}\right]$$

**Answer of Question 5**

a-i) This limit of indeterminate form $1^{\infty}$. Let $y=(\sin x)^{\tan x} \Rightarrow \ell n\, y=\tan x\, \ell n(\sin x)$ and we have to evaluate $\lim\limits_{x\to\frac{\pi}{2}} \ell n\, y = \lim\limits_{x\to\frac{\pi}{2}} \tan x\, \ell n(\sin x)$.

This intermediate form is $0.\infty$ and so rewrite the above limit such that

$\lim\limits_{x\to\frac{\pi}{2}} \tan x\, \ell n(\sin x) = \lim\limits_{x\to\frac{\pi}{2}} \frac{\ell n(\sin x)}{\cot x} = \frac{0}{0}$, then L'Hospital's rule can be used to evaluate this limit

$$\lim_{x\to\frac{\pi}{2}} \frac{\ell n(\sin x)}{\cot x} = \lim_{x\to\frac{\pi}{2}} \frac{\cos x/\sin x}{\csc^2 x} = \lim_{x\to\frac{\pi}{2}} \cos x \sin x = 0.$$

Therefore $\lim\limits_{x\to\frac{\pi}{2}} (\sin x)^{\tan x} = \lim\limits_{x\to\frac{\pi}{2}} e^{(\tan x)\,\ell n\,(\sin x)} = e^{\lim\limits_{x\to\frac{\pi}{2}} \tan x\, \ell n(\sin x)} = e^0 = 1$

$$\text{a-ii)} \lim_{x\to 0}\left[\frac{\sin x-\tan x}{x^3}\right]=\frac{0}{0} \Rightarrow \lim_{x\to 0}\left[\frac{\cos x-\sec^2 x}{3x^2}\right]=\lim_{x\to 0}\left[\frac{\cos^3 x-1}{3x^2\cos^2 x}\right]=\frac{0}{0}$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{-3\cos^2 x\ \sin x}{-6x^2\cos x\ \sin x+6x\cos^2 x}\right]=\frac{0}{0}$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{-3\cos^3 x+6\cos x\sin^2 x}{-24x\cos x\ \sin x+6x^2\sin^2 x-6x^2\cos^2 x+6\cos^2 x}\right]=-\frac{1}{2}$$

$$\text{a-iii)} \lim_{x\to 0}\left[\frac{1}{\ln(x+1)}-\frac{1}{e^x-1}\right]=\infty-\infty \Rightarrow \lim_{x\to 0}\left[\frac{(e^x-1)-\ln(x+1)}{[\ln(x+1)](e^x-1)}\right]=\frac{0}{0}$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{e^x-\frac{1}{(x+1)}}{[\ln(x+1)]e^x+\frac{(e^x-1)}{(x+1)}}\right]=\lim_{x\to 0}\left[\frac{(x+1)e^x-1}{(x+1)[\ln(x+1)]e^x+(e^x-1)}\right]=\frac{0}{0}$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{(x+1)e^x+e^x}{[\ln(x+1)]e^x+(x+1)[\ln(x+1)]e^x+2e^x}\right]=1$$

b) Let g(x) = $\ell n(x+1) \Rightarrow g`(x) = \dfrac{1}{x+1} \Rightarrow g``(x) = \dfrac{-1}{(x+1)^2} \Rightarrow g```(x) = \dfrac{2}{(x+1)^3}$

Therefore g(0) = 0, g`(0) = 1, g``(0) = -1, g```(0) = 2.

Hence $\ell n(x+1) = x - \dfrac{1}{2}x^2 + \dfrac{1}{3}x^3 + \ldots \Rightarrow x^3\ \ell n(x+1) = x^4 - \dfrac{1}{2}x^5 + \dfrac{1}{3}x^6 + \ldots$

c) Since f(x) is continuous on the interval [-7, -1], therefore mean value theorem is satisfied and so $f`(c) = \dfrac{f(-1)-f(-7)}{-1-(-7)} = \dfrac{-5-7}{6} = -2 = 3c^2 + 18c + 13 \Rightarrow c^2 + 6c + 5 = 0$

$\Rightarrow$ c = -5, -1

d -i) $\displaystyle\int \frac{\sec^2 3x}{\sqrt{1-\tan 3x}}dx = \frac{1}{3}\int (1+\tan 3x)^{-1/2}(3\sec^2 3x)\,dx = \frac{2}{3}\sqrt{(1+\tan 3x)} + c$

ii) $\displaystyle\int \frac{1}{x}(2+\frac{3}{x})^{-1}dx = \int [x\,[2+\frac{3}{x}]\ ]^{-1}\,dx = \frac{1}{2}\int [2x+3]^{-1}(\,2)dx = \frac{1}{2}\ell n(2x+3)+c$

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